2.2 The Continuity Equation

A basic principle of science and engineering is the conservation of mass. (Nuclear reactions will not be considered in these notes!) The continuity equation is an expression of this basic principle in a particularly convenient form for the analysis of materials processing operations.

Consider a stationary differential volume element of length Δx, width Δy and height Δz in a Cartesian coordinate system, as illustrated in Figure 2.2.

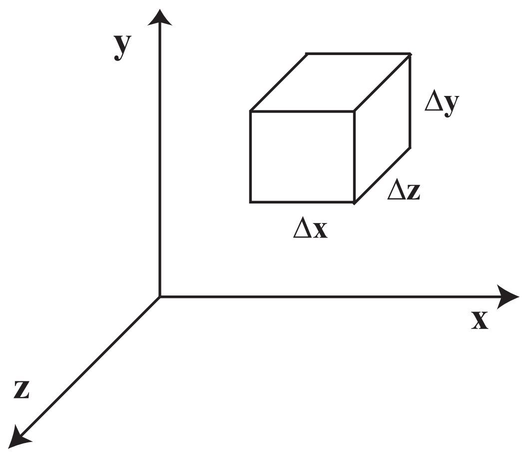

Figure 2.2: An arbitrary differential volume element.

The conservation of mass for this volume element ($\Delta V = \Delta x \Delta y \Delta z$) may be expressed verbally as:

Rate of change of mass in ΔV	=	Rate of mass convected into ΔV	-	Rate of mass convected out of ΔV .

Expressed mathematically this is:

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \Delta y \Delta z \left[(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x} \right] + \qquad (2.1)$$
$$\Delta x \Delta z \left[(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y} \right] + \Delta x \Delta y \left[(\rho v_z)|_z - (\rho v_z)|_{z+\Delta z} \right]$$

where ρ is the fluid density in ΔV. Dividing each side of the equation by ΔV, taking $\lim_{\Delta V \to 0}$, and invoking the definition of the partial derivative leads to:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right] . \tag{2.2}$$

This can be expressed more succinctly as:

$$\frac{\partial \rho}{\partial t} = -\underline{\nabla} \bullet (\rho \underline{\mathbf{v}}) \tag{2.3}$$

where $\underline{\mathbf{v}}$ is the velocity vector and $\underline{\nabla}$ is the "del" or "gradient" operator. In rectangular coordinates:

$$\underline{\nabla} = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} . \tag{2.4}$$

Equation 2.3 is the differential form of the continuity equation. A detailed explanation of the most commonly used coordinate systems can be found in Reference Notebook 1 while vector and tensor operations are explained in Reference Notebook 2. Rearrangement of Equation 2.3 leads to the equivalent expression:

$$\frac{D\rho}{Dt} = -\rho(\underline{\nabla} \bullet \underline{\mathbf{v}}) \tag{2.5}$$

where the substantial derivative $\frac{D}{Dt}$ is defined as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{\mathbf{v}} \bullet \underline{\nabla} . \tag{2.6}$$

Many fluids encountered in polymer processing operations are essentially incompressible: the fluid density is a constant. Thus ρ is a function of neither time nor space and the continuity equation reduces to:

$$\underline{\nabla} \bullet \underline{\mathbf{v}} = 0 . \tag{2.7}$$

The equation of continuity may be equivalently obtained in any appropriate coordinate system. Its expansions in the three most commonly used coordinate systems (rectangular, cylindrical, and spherical) are given in Tables 2.1 and 2.2.

In a similar manner, it is possible to derive the continuity equation over an arbitrary, spatially fixed closed region of macroscopic size. The enclosed volume is called the control volume, illustrated in Figure 2.3. Application of the conservation of mass to this volume leads to the integral form of the continuity equation:

Rectangular coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Spherical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0$$

Table 2.1: The equation of continuity in several coordinate systems.

Rectangular coordinates:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} = 0$$

Table 2.2: The equation of continuity for *fluids of constant density* in several coordinate systems.

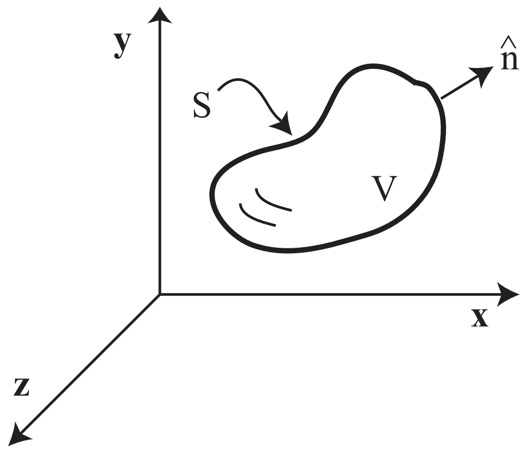

Figure 2.3: An arbitrary macroscopic control volume.

$$\frac{d}{dt}\int_V \rho dV = -\oint_S (\hat{\mathbf{n}} \bullet \rho\underline{\mathbf{v}})dS \;. \tag{2.8}$$

The equivalence of Equations 2.3 and 2.8 may be easily demonstrated through use of the divergence theorem. The divergence theorem of Gauss states that if V is a volume bounded by a closed surface S and $\underline{\mathbf{A}}$ is a continuous vector field, then:

$$\int_V (\underline{\nabla} \bullet \underline{\mathbf{A}})dV = \oint_S (\hat{\mathbf{n}} \bullet \underline{\mathbf{A}})dS \;. \tag{2.9}$$

Thus the surface integral in Equation 2.8 may be converted to a volume integral:

$$\frac{d}{dt}\int_V \rho dV = -\int_V (\underline{\nabla} \bullet \rho\underline{\mathbf{v}})dV \;. \tag{2.10}$$

Since the control volume is fixed in space, the ordinary derivative may be brought inside of the integral and changed into a partial derivative, allowing both sides of the equation to be consolidated within the integral:

$$\int_V \left[\frac{\partial\rho}{\partial t} + (\underline{\nabla} \bullet \rho\underline{\mathbf{v}})\right] dV = 0 \;. \tag{2.11}$$

Because this equation holds for an arbitrary volume V, the integrand must vanish, leading the the same expression which was derived from the infinitestimally small volume element:

$$\frac{\partial \rho}{\partial t} + (\underline{\nabla} \bullet \rho \underline{\mathbf{v}}) = 0 \ . \tag{2.12}$$

2.2.1 Example: The Freely Rising Foam

A block of foam is produced by pouring a reacting mixture into the bottom of a trough. The chemical reaction produces a blowing agent which reduces the density of the fluid according to the equation:

$$\rho = \rho_f + (\rho_0 - \rho_f)e^{-t/\tau} \ . \tag{2.13}$$

At any given time, the density of the material is constant throughout the sample. The trough is sufficiently large that the walls have no effect on the rise of the foam. The trough is initially filled to a height of H_0 by the reacting liquid. What is the velocity distribution within the sample as a function of time and position within the rising sample? What is the final height of the foam?

Solution: Spatially, this is clearly a one-dimensional problem, with the critical coordinate in the direction of rise of the foam. The continuity equation reduces immediately to:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho v_z}{\partial z} \ . \tag{2.14}$$

Since the density is a function of time but independent of position, the left hand side may be changed to an ordinary derivative and the density may be removed from the partial derivative on the right hand side,

$$\frac{d\rho}{dt} = -\rho \frac{\partial v_z}{\partial z} \ . \tag{2.15}$$

The left hand side may be evaluated directly from the equation given for the kinetics of density evolution. Some rearrangement leads to:

$$\frac{\partial v_z}{\partial z} = \frac{\frac{\rho_0 - \rho_f}{\tau} e^{-t/\tau}}{\rho_f + (\rho_0 - \rho_f)e^{-t/\tau}} = \beta(t) \tag{2.16}$$

where the function $\beta(t)$ has been defined for convenience. This may be integrated directly to give:

$$v_z = \beta(t) z + g(t) \tag{2.17}$$

where $g(t)$ is an unknown function of time which is to be determined by a boundary condition. The material at the very bottom of the trough ($z = 0$) never rises off of the bottom so that the boundary condition is $v_z(z = 0, t) = 0$. In order for this to be the case, it is necessary that $g(t) = 0$. The velocity distribution is thus given by:

$$v_z = \beta(t) z \ . \tag{2.18}$$

The final height of the film is easily obtained through a macroscopic mass balance. Since the total mass in the system is conserved throughout the process, we have:

$$mass = \rho_0 H_0 (Area) = \rho_f H_f (Area) \tag{2.19}$$

where $Area$ is the cross-sectional area of the trough. This immediately provides:

$$H_f = H_0 \frac{\rho_0}{\rho_f} \ . \tag{2.20}$$

2.2.2 Problems

1. **The Incompressibility Assumption.** In the analysis of polymer processing operations, it is often assumed that the polymer melt is incompressible. Is it accurate to assume that the density of an isothermal polymer melt is constant? Support your answer with data.

2. **Slot Coating.** Consider a slot coating operation where an incompressible fluid is coated onto a moving web, as illustrated in Figure 2.4. The fluid to be coated is pumped through a slot die at a constant volumetric flow rate Q. Both the slot die and the web have depth W (into the page). The web moves at a constant velocity V. The gap height between the bottom of the die and the top of the web is H_0.

   (a) What is the final thickness of the coating on the web, H_∞?

   (b) On a vertical plane between the die and the web at a point to the left of the injection slot, what is the net volumetric flow rate of fluid? What potential problem with the slot coating operation does this analysis suggest?

   (c) On a vertical plane between the die and the web at a point to the right of the injection slot, what is the net volumetric flow rate of fluid?

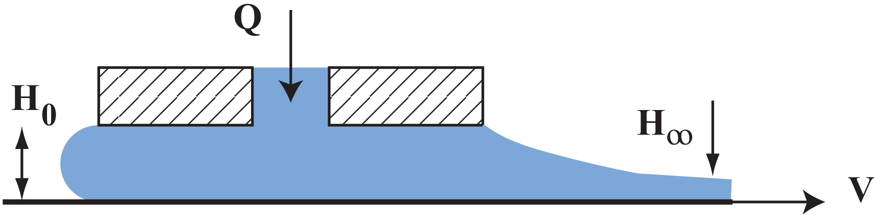

Figure 2.4: Schematic of a slot coating operation.

3. **Film Casting.** A thin polymer film of width W is prepared using film casting, as illustrated in Figure 2.5 in cross-section. A melt of density ρ_L flows through a die of width W and slot thickness H_O at a volumetric flow rate of Q. The melt film is taken up by the chill roll of radius R which rotates at an angular velocity ω. The polymer solidifies on the wheel, and is later stripped off in film form. The density of the solidified polymer is ρ_S. The process is operated at steady state. The radius of the chill roll is much larger than the thickness of the film, $R >> H_\infty$.

   (a) What is the final thickness of the film produced, H_∞?

   (b) In the region where both molten and solid portions of the polymer exist (such as at cross section A in the close-up), how is the rate of increase in the thickness of the solidified portion of the film related to the rate of decrease in the total thickness of the film on the wheel? In this region, you may assume that the horizontal velocity of both the molten and solid portions of the film are the same as that of the rotating wheel.

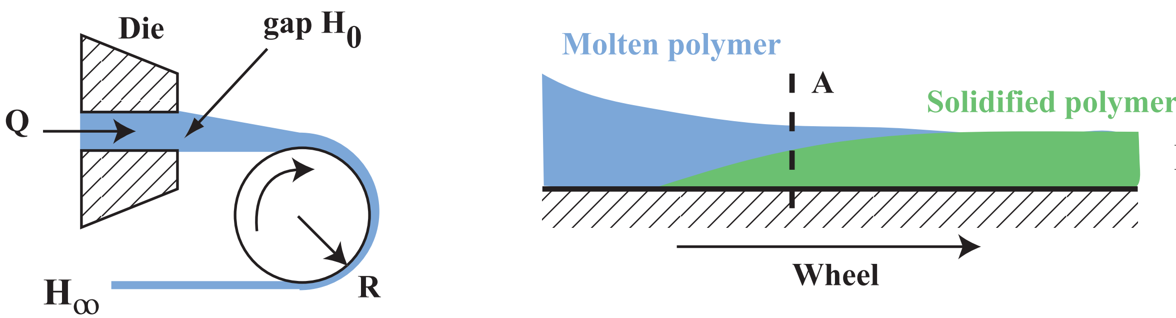

Figure 2.5: Schematic of a film casting operation. A close-up of the solidification zone is given at right.