2.3 The Equation of Motion

The equation of motion is a particularly useful expression of the principle of conservation of momentum. Our analysis of conservation of mass, a scalar quantity, in the previous section resulted in a vector equation. We may thus expect that our analysis of conservation of momentum, a vector quantity, will result in a tensor equation. Tensor definitions and mathematical operations are described in the Operations with Vectors and Tensors Supplementary Notes.

2.3.1 Conservation of Momentum

We consider once again the arbitrary macroscopic volume element, Figure 2.6, which is reproduced below. Momentum may be transported into this control volume through convection: the bulk flow of fluid across the surface. It may also be transported into the control volume through forces which act on its surfaces. In addition, body forces acting on the material in the volume change its momentum. We can express these various possibilities verbally as:

Rate of change of momentum in V	=	Net rate of momentum convected into V	+	Net rate of momentum creation by surfaces forces on V	+	Net rate of momentum creation by body forces on V .

The local volumetric rate of flow of fluid across a surface element dS is $(\hat{\mathbf{n}} \bullet \underline{\mathbf{v}})dS$. Thus, the rate of momentum convection due to flow of fluid across the control volume surface is $(\hat{\mathbf{n}} \bullet \underline{\mathbf{v}})\rho\underline{\mathbf{v}}dS$; which can be rearranged to $(\hat{\mathbf{n}} \bullet \rho\underline{\mathbf{v}}\,\underline{\mathbf{v}})dS$.

There will also be momentum transferred through the surface of the control volume due to the molecular motions and interactions within the fluid itself. Thus, we need something that represents the flux of momentum across the surface of the control volume due to these effects. This momentum flux may be obtained from the total stress tensor, $\underline{\underline{\pi}}$. The component π_{ij} represents the flux of positive j-momentum in the positive i-direction. The rate of flow of momentum across a differential surface element of area dS and orientation $\hat{\mathbf{n}}$ is $(\hat{\mathbf{n}} \bullet \underline{\underline{\pi}})dS$. We can thus think of the dot product $(\hat{\mathbf{n}} \bullet \underline{\underline{\pi}})dS$ as a machine that gives the flow rate of momentum across a surface due to

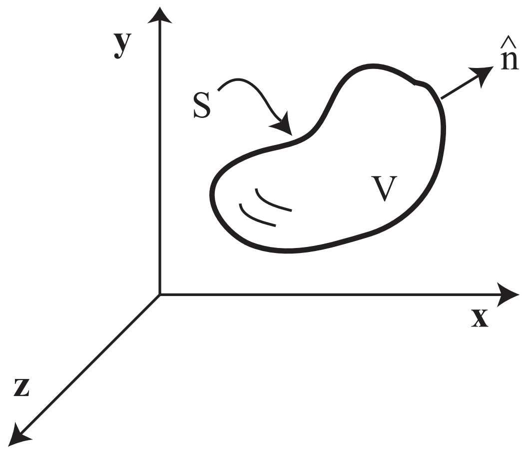

Figure 2.6: An arbitrary macroscopic control volume.

molecular motion effects within the fluid.

There is also an alternative interpretation of the total stress tensor $\underline{\underline{\pi}}$ and its components. The total momentum of the fluid within V will change due to forces on the surface of the body. This surface forces term in our macroscopic balance is then:

$$-\oint_S \left(\hat{\mathbf{n}} \bullet \underline{\underline{\pi}}\right) dS \tag{2.21}$$

where $(\hat{\mathbf{n}} \bullet \underline{\underline{\pi}})dS = \underline{\pi}_n dS$ is a vector that describes the force exerted by the fluid on the negative side of dS onto the fluid on the positive side of dS as illustrated in Figure 2.7.

The reader should be aware of a different convention with regard to the total stress tensor. In applied mechanics and mechanical engineering a stress tensor $\underline{\underline{\sigma}}$ is commonly used, where $\underline{\underline{\pi}} = -\underline{\underline{\sigma}}^T$. With the convention described in these notes, compressive pressure is positive and tension is negative. Thus, this pressure is the same as the pressure encountered in the study of thermodynamics. With the $\underline{\underline{\sigma}}$ convention tensile stresses are positive and compressive forces are negative. (Why would this convenient in applied mechanics?)

Body forces may be present due to a variety of effects, such as gravitational, electrical, or magnetic. Traditionally, these are represented in a generic balance simply as $\rho\underline{\mathbf{g}}$. Other forces may be substituted as the particular problem requires.

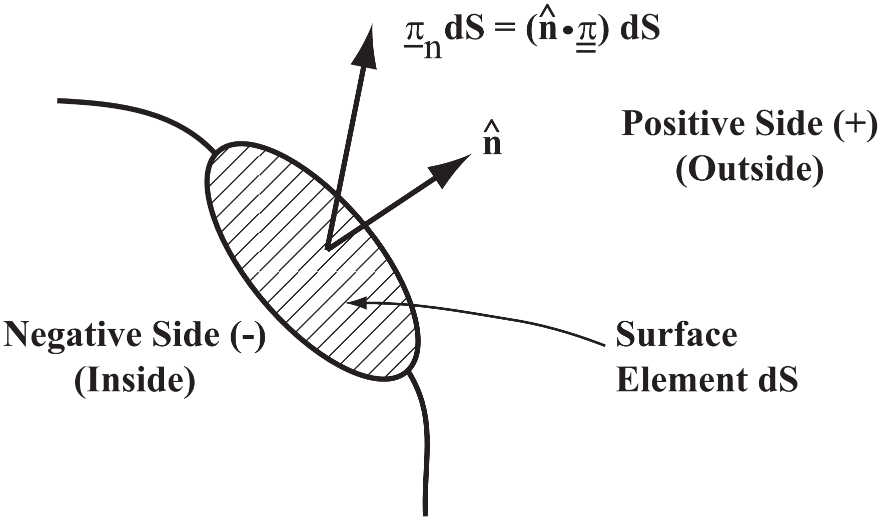

Figure 2.7: Physical significance of the stress tensor at a differential surface element.

Thus, the momentum balance which was described in words can now be stated mathematically as:

$$\frac{d}{dt}\int_V \rho\underline{\mathbf{v}}dV = -\oint_S(\hat{\mathbf{n}}\bullet\rho\underline{\mathbf{v}}\,\underline{\mathbf{v}})dS - \oint_S\left(\hat{\mathbf{n}}\bullet\underline{\underline{\pi}}\right)dS + \int_V \rho\underline{\mathbf{g}}dV \tag{2.22}$$

where $\underline{\underline{\pi}}$ is the total stress tensor and $\underline{\mathbf{g}}$ is the sum of the body forces present in the volume. Since the volume V is arbitrary, the Gauss divergence theorem may be used to transform this integral to a differential form of the equation of motion:

$$\frac{\partial}{\partial t}(\rho\underline{\mathbf{v}}) = -[\underline{\nabla}\bullet\rho\underline{\mathbf{v}}\,\underline{\mathbf{v}}] - \left[\underline{\nabla}\bullet\underline{\underline{\pi}}\right] + \rho\underline{\mathbf{g}}\,. \tag{2.23}$$

It is conventional for the total stress tensor $\underline{\underline{\pi}}$ to be divided into two parts:

$$\underline{\underline{\pi}} = P\underline{\underline{\delta}} + \underline{\underline{\tau}} \tag{2.24}$$

where P is a scalar called the hydrostatic pressure, $\underline{\underline{\tau}}$ is the "stress tensor," and $\underline{\underline{\delta}}$ is the unit or identity tensor,

$$\underline{\underline{\delta}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.25}$$

Note that in an incompressible fluid, the absolute value of the pressure is arbitrary. Thus, one may select the value of a particular pressure such as a boundary condition in order to simplify solution of the problem. In an incompressible fluid, it is *changes* in pressure (in time and/or space) that contribute to flows, not the absolute pressure itself.

Based on the definition of the stress tensor, an equivalent form of the equation of motion equivalent to 2.23 is:

$$\frac{\partial}{\partial t}(\rho \underline{\mathbf{v}}) = -[\underline{\nabla} \bullet \rho \underline{\mathbf{v}}\, \underline{\mathbf{v}}] - \underline{\nabla} P - \left[\underline{\nabla} \bullet \underline{\underline{\tau}}\right] + \rho \underline{\mathbf{g}} \, . \tag{2.26}$$

With the aid of the continuity equation and the definition of the substantial derivative, this may be transformed to:

$$\rho \frac{D\underline{\mathbf{v}}}{Dt} = -\underline{\nabla} P - \left[\underline{\nabla} \bullet \underline{\underline{\tau}}\right] + \rho \underline{\mathbf{g}} \tag{2.27}$$

which is the form in which the equation of motion is generally expressed. The terms on the left-hand side of the equation are called the "inertial terms." The physical significance of the remaining terms is evident based on the qualitative discussion of mechanisms given above. The expanded form of the equation of motion in several coordinate systems is given in Tables 2.3, 2.4, and 2.5 at the end of this section.

Due to the high viscosities of polymers, polymer processing flows are often creeping flows. This is a flow in which the viscous forces predominate over the inertial forces. In this case, the equation of motion reduces to:

$$\rho \frac{\partial \underline{\mathbf{v}}}{\partial t} = -\underline{\nabla} - \left[\underline{\nabla} \bullet \underline{\underline{\tau}}\right] + \rho \underline{\mathbf{g}} \, . \tag{2.28}$$

Examples of these flows include those treated by the lubrication approximation, Hele-Shaw flows, and the flow of very viscous fluids past immersed bodies.

2.3.2 Constitutive Equations

Thus far, we have considered application of the principle of conservation of momentum to the solution of polymer processing flow problems, with little regard for the properties of the fluid or material itself. Consider carefully Equation 2.27. Given appropriate initial and boundary conditions, what additional information will be needed in order to solve for the velocity field as a function of space and time? We require information that relates the stress tensor $\underline{\underline{\tau}}$ to the velocity field. In physical terms, we must be able to relate the stresses in the material to its deformation and/or rate of deformation. For a perfectly elastic solid, such as a Hookean material, the stresses are related to the deformation. For a completely viscous fluid, such as a Newtonian liquid, the stresses are related to the rate of deformation. For materials which exhibit intermediate behavior, such as viscoelastic fluids, we may need to consider both the magnitude and rate of deformation.

The stresses in many materials during polymer processing operations can be accurately related to their rate of deformation, as quantified by the rate of deformation tensor $\underline{\underline{\dot{\gamma}}}$. This tensor is constructed from gradients of the velocities. These gradients are captured by the velocity gradient tensor, $\underline{\nabla}\,\underline{\mathbf{v}}$, which is the dyadic product of $\underline{\nabla}$ and $\underline{\mathbf{v}}$. (The dyadic product is defined in the Operations with Vectors and Tensors supplementary notes.) The velocity gradient tensor may be decomposed into two parts:

$$\underline{\nabla}\,\underline{\mathbf{v}} = \frac{1}{2}\left(\underline{\underline{\dot{\gamma}}} + \underline{\underline{\omega}}\right) \tag{2.29}$$

where $\underline{\underline{\dot{\gamma}}}$ is the rate of deformation (also called rate of strain) tensor and $\underline{\underline{\omega}}$ is the vorticity tensor. These are each defined as

$$\underline{\underline{\dot{\gamma}}} = \underline{\nabla}\,\underline{\mathbf{v}} + \left(\underline{\nabla}\,\underline{\mathbf{v}}\right)^T \tag{2.30}$$

$$\underline{\underline{\omega}} = \underline{\nabla}\,\underline{\mathbf{v}} - \left(\underline{\nabla}\,\underline{\mathbf{v}}\right)^T \tag{2.31}$$

where the superscript T indicates the transpose of the tensor. The vorticity tensor is concerned primarily with rotational motions which are not associated with deformation of a body.

For simple shearing flows, such as steady shear between infinite parallel plates indicated in Figure 2.8, $\underline{\underline{\dot{\gamma}}}$ reduces to:

$$\underline{\underline{\dot{\gamma}}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma} \tag{2.32}$$

where $\dot{\gamma}$ is the shear rate. The shear rate is a scalar related to the second invariant of the rate of deformation tensor:

$$\dot{\gamma} = \sqrt{\frac{1}{2}\left(\underline{\underline{\dot{\gamma}}}:\underline{\underline{\dot{\gamma}}}\right)} \tag{2.33}$$

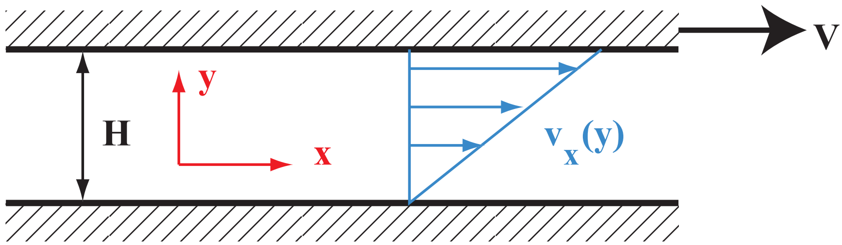

Figure 2.8: Simple shear flow between infinite parallel plates.

Now that we have quantitative expressions for the stresses $\left(\underline{\underline{\tau}}\right)$ and rates of deformation $\left(\underline{\underline{\dot{\gamma}}}\right)$ within a material, we may write constitutive relationships in equation form. The simplest fluid is a Newtonian fluid of constant viscosity, for which:

$$\underline{\underline{\tau}} = -\mu \underline{\underline{\dot{\gamma}}} \tag{2.34}$$

where μ is the shear viscosity. For a Newtonian fluid of constant viscosity and constant density, the equation of motion reduces to:

$$\rho \frac{D\underline{\mathbf{v}}}{Dt} = -\underline{\nabla} P - \underline{\nabla}^2 \underline{\mathbf{v}} + \rho \underline{\mathbf{g}} \tag{2.35}$$

which is the Navier-Stokes equation. $\underline{\nabla}^2$ is, of course, the Laplacian operator. This equation is given in expanded form in the three primary coordinate systems in Tables 2.6, 2.7 and 2.8 at the end of this section.

Another important type of fluid is the power law fluid. The constitutive relation for a power law fluid is:

$$\underline{\underline{\tau}} = -m \dot{\gamma}^{n-1} \underline{\underline{\dot{\gamma}}} \tag{2.36}$$

and thus the viscosity of this fluid is:

$$\eta = m \dot{\gamma}^{n-1} \tag{2.37}$$

where m is the power law prefactor and n is the power law exponent. The parameter m is usually regarded as a function of temperature. Data on many polymer melts indicates that m often obeys an Arrhenius type of relationship so that:

$$m = m_0 \exp \frac{\Delta E}{R} \left(\frac{1}{T} - \frac{1}{T_0} \right) \tag{2.38}$$

where m_0 is the value of m at T_0, ΔE is the flow activation energy, R is the gas constant, and the temperature must be expressed in degrees Kelvin.

2.3.3 Boundary Conditions

In addition to the conservation and constitutive equations, initial and boundary conditions are required in order to solve a flow problem.

1. **Solid-Fluid Interfaces**

   (a) *The Slip Condition.* By far, the most common assumption at solid-fluid boundaries is the no-slip assumption, which asserts simply that there is no relative motion between the two phases at the interface:

   $$\underline{\mathbf{v}}_{fluid} = \underline{\mathbf{v}}_{solid} \text{ at the boundary .} \tag{2.39}$$

   There are conditions under which it is believed that slip takes place between a fluid and solid, especially at high shear rates. Slip at the wall is believed to occur at high flow rates, for example, in the melt fracture regime of die extrusion. The melt fracture may in fact be due to an intermittent slip-stick phenomena related to the viscoelastic nature of the material. Various different models for interfacial slip have appeared in the literature.

   (b) *Solid Penetrability.* In the vast majority of cases, the fluid is assumed not to penetrate the surface of the solid. The equation given immediately above accurately implements this. If there is penetration of the solid by the fluid, due perhaps to porosity, the fluid velocity at the interface is usually separated into normal and tangential components.

2. **Fluid-Fluid Interfaces** In this case there are four types of boundary conditions that must be specified.

   (a) *The Slip Condition.* The same no-slip assumption described above is typically assumed for fluid-fluid interfaces. This means that there is continuity of the tangential velocities at the interface.

   $$\underline{\mathbf{v}}_{fluidA} = \underline{\mathbf{v}}_{fluidB} \text{ at the boundary .} \tag{2.40}$$

   There are exceptions to its applicability.

(b) *Interface Penetrability.* In the vast majority of cases, the fluids are assumed not to penetrate across the interface. Similar comments apply as for the fluid-solid interface. The equation given directly above implements in impenetrable boundary condition, establishing the continuity of normal velocities at the interface.

(c) *Balance of Shear Stresses at the Interface.* The assumption is usually made that the shear stresses are continuous across the interface:

$$\underline{\underline{\mathcal{T}}}_{fluidA} = \underline{\underline{\mathcal{T}}}_{fluidB} \text{ at the boundary .} \tag{2.41}$$

There are very few exceptions to this. One notable one would be the case where the interface itself is considered a membrane with some elastic properties.

(d) *Balance of Normal Stresses and Interfacial Tension across the Interface.* The normal stresses are, in general, not continuous across the interface. They differ according to:

$$P_B - P_A = \Gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{2.42}$$

where $P_B - P_A$ is the pressure difference due to the interfacial or surface tension Γ acting on a curved surface with principal radii of curvature R_1 and R_2. The sign convention for R_1 and R_2 is as follows: $R_i > 0$ if the corresponding center of curvature lies on the B side of the interface. Therefore the pressure is higher in the fluid on the concave side of the interface. In polymer melt processing the effects of interfacial tension are often negligible. If the interfacial tension is negligible or zero, then the pressure is the same on both sides of the interface.

Consider the three situations illustrated in Figure 2.9. Figure 2.9a illustrates a sphere of radius R of component B inside a matrix of component A. In this case, both principal radii of curvature are R and the centers of curvature are both inside B. Thus,

$$P_B - P_A = \frac{2\Gamma}{R} \tag{2.43}$$

and the pressure is higher in the B domain. Exchange the labels on the components and rework the problem to obtain the same answer to demonstrate the significance of the sign convention. Figure 2.9b illustrates a cylinder of radius R and infinite length of component B inside a matrix of component A. In this case, one principal radii of curvature is R and the other is ∞. Thus, the pressure inside the cylinder is Γ/R larger than that outside. Figure 2.9c illustrates a B domain of complex shape inside a A matrix. In this case, there are no global principal radii of curvature; instead, the radii are potentially different at each location on the interface. Thus, the pressure difference across the interface due to interfacial tension is potentially different at each location. What implications does this have for interfacial tension generated flows? What implications does this have for the solution of a flow problem with unusually shaped domains?

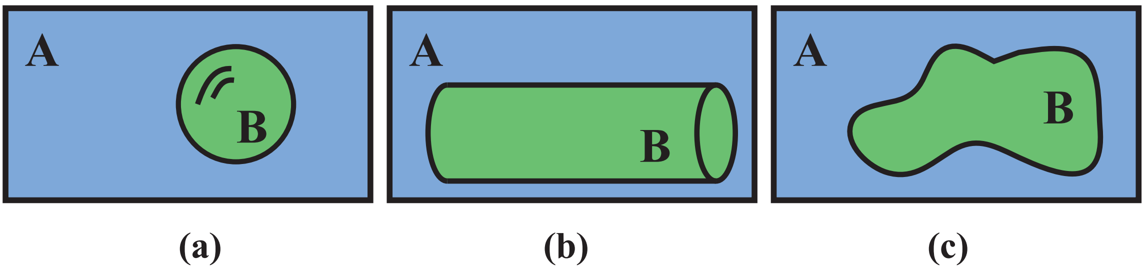

Figure 2.9: Various shapes of a fluid domain inside a matrix.

2.3.4 Example: Die Flow in Film Casting

Casting is commonly used to manufacture films or sheets of thermoplastics such as polyolefins, polyesters, and nylons. A single screw extruder melts the polymer and builds the pressure required to pump it through a die in order to establish the geometry of the film or sheet. The molten material is then cooled in its final form. The cross-section of a simple film die is shown in Figure 2.10. This figure approximates the die flow as a pressure driven between parallel plates. The pressure difference responsible for the flow is the difference between the pressures at the entrance to and exit from the die, $\Delta P = P_2 - P_1$. We are interested specifically in the relationship between this pressure difference and the flow rate of polymer through the die. The flow rate is, of course, closely related to the production rate of the manufacturing operation. In addition, we may be interested in the velocity field so that we can understand the stresses and deformation rates that the material experiences as it moves through the process. Designers of the die itself will want to know the forces exerted by the flowing polymer on the die to ensure that it is mechanically robust enough the contain the flow without deforming.

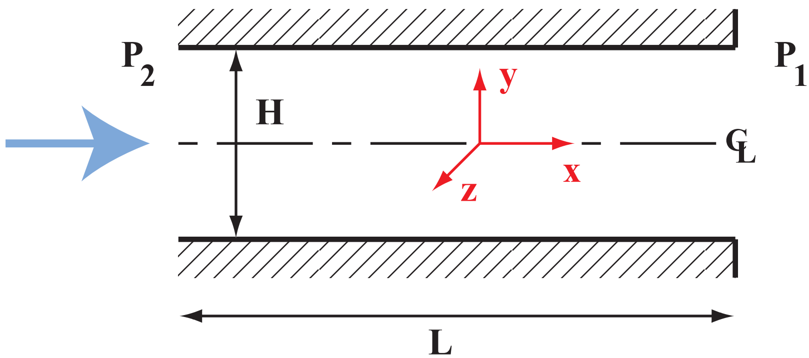

Figure 2.10: Cross-section of a film casting die.

Consider the well developed, steady-state pressure driven flow of an incompressible power law fluid through the die. By describing the flow as "well developed," we mean that we will consider the flow to be that between parallel plates throughout the entire flow regime under consideration. In actuality, there will be some rearrangement of the flow as it enters the die from the extruder; there will also be some disruption of the flow as it exits the end of the die into the open air. However, these entrance and exit effects are neglected here. By describing the flow as "steady-state," we mean that all aspects of the flow are time independent. Film casting manufacturing operations are run as close to steady-state as is feasible in order to produce a uniform film thickness; however, there are small time dependent fluctuations in even the best of production lines.

These notes will also often make the (unstated) assumption that the flow under consideration is laminar and well ordered. This means that the material flow can be considered as being comprised of layers of fluid which slide past each other. (Imagine these layers as individual playing cards if you were shearing a deck of cards.) The counterpart of laminar flow is turbulent flow. However, the high viscosities of polymer melts nearly always prevent turbulence. The transition between laminar and turbulent flow in a flow between parallel plates is governed by the Reynolds number:

$$Re = \frac{HV\rho}{\mu} \tag{2.44}$$

where V is a representative velocity. A typical film casting operation might have a Reynolds number around 0.001, very much below that required for transition to turbulence.

Clearly the rectangular coordinate system is most convenient for analysis of this problem. We may immediately conclude from the symmetry of the problem that there is no dependence of any variable on the z-coordinate, all $\frac{\partial}{\partial z} = 0$. In addition, because this is a steady-state flow, all $\frac{\partial}{\partial t} = 0$. It is also clear that $v_z = 0$, $v_y = 0$, and in terms of the velocities we are interested only in $v_x = v_x(y)$. (Why? Check these conclusions for consistency using the continuity equation.) Since the effect of gravity is negligible here, we also have $P = P(x)$ only.

On the basis of the results above, the x-component of the equation of motion reduces to:

$$0 = -\frac{dP}{dx} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right) . \tag{2.45}$$

One must be careful to choose carefully between the general form of the equation of motion (Equation 2.27) and the Navier-Stokes equation (Equation 2.35). Since this problem concerns a power law fluid, the general form must be used. The simplicity of the velocity field allows easy construction of the components of the stress tensor. The velocity gradient tensor

$$\underline{\nabla}\,\underline{\mathbf{v}} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{dv_x}{dy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.46}$$

is not nearly so frightening when eight of its nine components are zero. Note also the use of ordinary derivatives instead of partial derivatives when it becomes clear that this is appropriate. Thus, the rate of deformation tensor is:

$$\underline{\underline{\dot{\gamma}}} = \begin{bmatrix} 0 & \frac{dv_x}{dy} & 0 \\ \frac{dv_x}{dy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{2.47}$$

Based on the constitutive equation for a power law fluid (Equation 2.36) and the definition of the shear rate (Equation 2.33), we obtain $\tau_{xx} = 0$ and

$$\tau_{xy} = \tau_{yx} = -m\dot{\gamma}^{n-1}\frac{dv_x}{dy} \tag{2.48}$$

where

$$\dot{\gamma} = \left|\frac{dv_x}{dy}\right| . \tag{2.49}$$

The absolute value has been used because the shear rate in Equation 2.33 is specifically the positive square root.

The equation of motion may thus be rearranged to:

$$\frac{dP}{dx} = m\frac{d}{dy}\left[\dot{\gamma}^{n-1}\frac{dv_x}{dy}\right] . \tag{2.50}$$

The most direct procedure for solution of this equation is to recognize that the left hand side is a function of x only while the right hand side is a function of y only. In order for the equality to hold throughout the entire flow domain, each side must be equal to some constant, C_1. Thus,

$$\frac{dP}{dx} = C_1 . \tag{2.51}$$

Integration and application of the appropriate boundary conditions yields an explicit expression for the pressure:

$$P(x) = P_2 - \Delta P\frac{x}{L} . \tag{2.52}$$

This result may now be substituted into the other side of the original differential equation:

$$m\frac{d}{dy}\left[\dot{\gamma}^{n-1}\frac{dv_x}{dy}\right] = -\frac{\Delta P}{L} \tag{2.53}$$

allowing another integration:

$$\left|\frac{dv_x}{dy}\right|^{n-1}\frac{dv_x}{dy} = -\frac{\Delta P}{mL}y + C_2 \tag{2.54}$$

In order to simplify the solution, we will now take advantage of the symmetry of the problem and limit our analysis to the regime $y \leq 0$. Since the velocity is zero at the wall (no-slip condition) and is expected to be a maximum at the center line, we certainly expect $\frac{dv_x}{dy} \geq 0$ in this regime and the absolute value signs may be dropped:

$$\left(\frac{dv_x}{dy}\right)^n = -\frac{\Delta P}{mL} y + C_2 \, . \tag{2.55}$$

Due to symmetry, $\frac{dv_x}{dy} = 0$ at $y = 0$ so that $C_2 = 0$. Rearranging and normalizing the y coordinate by $H/2$ gives:

$$\frac{dv_x}{dy} = \left(\frac{H}{2}\frac{\Delta P}{mL}\right)^{1/n} \left(-\frac{2y}{H}\right)^{1/n} \, . \tag{2.56}$$

Integration of this equation and application of the no-slip boundary condition ($v_x(y = -H/2) = 0$) to determine the constant of integration provides an explicit expression for the velocity field:

$$v_x = \frac{nH}{2(1+n)} \left(\frac{H\Delta P}{2mL}\right)^{1/n} \left[1 - \left(\frac{-2y}{H}\right)^{\frac{1}{n}+1}\right] \, . \tag{2.57}$$

Expanding this result over the entire flow regime leads to the final result:

$$v_x = \frac{nH}{2(1+n)} \left(\frac{H\Delta P}{2mL}\right)^{1/n} \left[1 - \left|\frac{2y}{H}\right|^{\frac{1}{n}+1}\right] \, . \tag{2.58}$$

This result is most conveniently illustrated using the dimensionless variables v_x^* and y^* defined as follows:

$$v_x^* = \frac{v_x}{v_x^{max}} \tag{2.59}$$

$$y^* = 2\frac{y}{H} \tag{2.60}$$

where v_x^{max} is the maximum velocity (at the center of the channel). This dimensionless velocity field is plotted in Figure 2.11 for a number of values of the power law index. For the case of $n = 1$ we have the familiar parabolic profile of a Newtonian fluid. As the power law index decreases, the velocity profile becomes more and more flattened in the center. Many commercial polymer melts have a power law index in the range $0.3 \leq n \leq 0.7$. This flattening of the velocity profile near the channel center and relegation of the

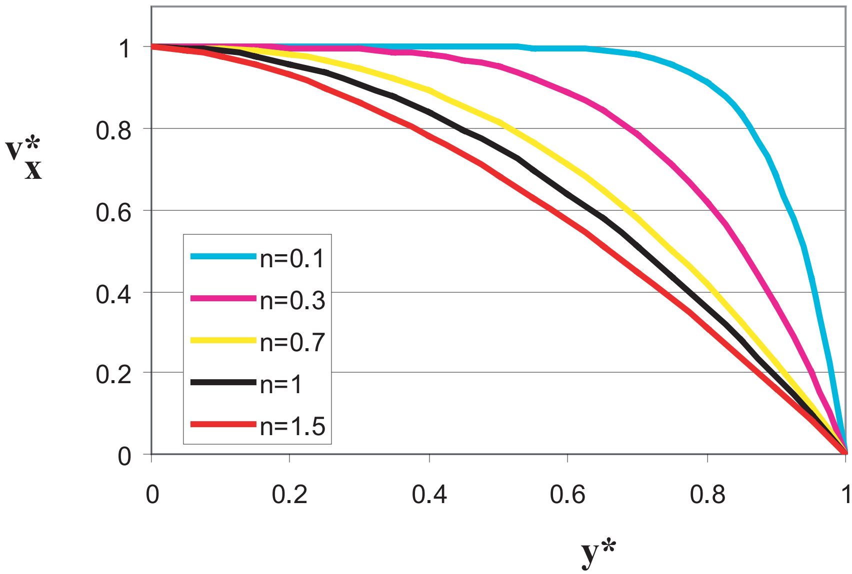

Figure 2.11: Dimensionless velocity distribution as a function of position and power law index for pressure driven flow of a power law fluid between parallel plates.

majority of shear toward the walls has numerous implications for polymer processing flows and heat transfer.

The relationship between the pressure drop and flow rate is obtained by integration of the velocity across the height of the flow regime. The volumetric unit per unit width

$$\frac{Q}{W} = \int_{-H/2}^{H/2} v_x dy \tag{2.61}$$

where W is the width of the die (into the cross-section of Figure 2.10). The result of this integration is simply provided here:

$$\frac{Q}{W} = \frac{H^2}{2\left(\frac{1}{n}+2\right)} \left(\frac{H\Delta P}{2mL}\right)^{1/n} . \tag{2.62}$$

The shear rate varies across the channel, however, the nominal shear rate for this type of flow is usually taken to be the shear rate at the wall:

$$\left.\frac{dv_x}{dy}\right|_{y=H/2} = \left(\frac{H}{2}\frac{\Delta P}{mL}\right)^{1/n} . \tag{2.63}$$

In addition, we are interested in the force/unit area exerted by the flowing polymer melt on the walls of the channel. There are several methods of calculating this; however, we will use this example to illustrate use of the total stress tensor for this purpose. All of the components of $\underline{\underline{\pi}}$ have already been determined:

$$\underline{\underline{\pi}} = \begin{bmatrix} P(x) & -m\left|\frac{dv_x}{dy}\right|^{n-1}\frac{dv_x}{dy} & 0 \\ -m\left|\frac{dv_x}{dy}\right|^{n-1}\frac{dv_x}{dy} & P(x) & 0 \\ 0 & 0 & P(x) \end{bmatrix} \tag{2.64}$$

since explicit expressions for the pressure and velocity are known. Because we are interested in the force exerted by the flowing fluid upon the die wall, we select $\hat{\mathbf{y}}$ as the outward pointing normal, Figure 2.12. The force per unit area exerted by the fluid on the wall is simply (just like in Figure 2.7)

$$\hat{\mathbf{y}} \bullet \underline{\underline{\pi}} = \left[-m\left|\frac{dv_x}{dy}\right|^{n-1}\frac{dv_x}{dy}\right]\hat{\mathbf{x}} + [P(x)]\,\hat{\mathbf{y}} + [0]\,\hat{\mathbf{z}} . \tag{2.65}$$

The $\hat{\mathbf{x}}$ term is the shear force, the $\hat{\mathbf{y}}$ term is the normal force, and there is

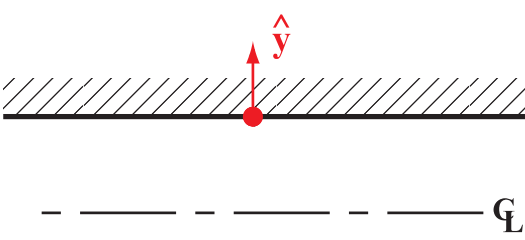

Figure 2.12: Selection of the outward pointing normal for calculation of forces on the die.

no force in the z-direction. Are the signs of the terms consistent with the directions of the forces you expect? How does the answer change if we select

$-\hat{\mathbf{y}}$ as the outward pointing normal at the lower lip of the die? How would you obtain the total force on the die surfaces rather than just the force per unit area? How would the approach be different if we were to calculate the force exerted by the die surface on the polymer melt? Can you see a faster way to calculate these forces based on the simple geometry of this particular problem?

2.3.5 Example: Blowing of a Cylindrical Parison

Blow molding is a shaping operation commonly used with amorphous polymers and rapidly crystallizing semicrystalline polymers. A hot cylindrical parison is inflated against a cold mold in order to form a hollow object such as a bottle. In the central region of the object, the process can often be accurately approximated as the inflation of a cylindrical shell.

Consider the inflation of a cylindrical shell of polymer melt, as illustrated in Figure 2.13. The cylindrical shell is initially of radius R_0 and wall thickness H_0. Starting at time zero, it is inflated by the application of air pressure $P_1(t)$ at the center of the cylindrical "bubble." Subsequently, the radius $R(t)$ increases as a function of time and the wall thickness $H(t)$ decreases. Throughout the process, assume that the cylindrical shape is maintained and the length of the cylinder is constant at L. At all times the radius is much larger than the wall thickness, $R(t) \gg H(t)$; in addition, the length of the cylinder is much larger than the radius, $L \gg R(t)$. The pressure on the outside of the cylindrical shell is constant at P_{atm}. The polymer contacts the mold wall at the final radius R_f. The density of the polymer melt is constant.

As the cylindrical shell is inflated, the pressure $P_1(t)$ is adjusted so that the inflation velocity $dR(t)/dt$ is constant throughout at a value of V_0. What is $R(t)$? What is $H(t)$? What is the velocity distribution within the expanding cylindrical shell of polymer? What type of flow is this? What is the deformation rate as a function of time?

Assume that the polymer melt is a Newtonian fluid of constant viscosity μ. What is the blowing pressure $P_1(t)$ required to produce the specified deformation? [Hint: inertial effects and interfacial tension are negligible.]

Suppose instead that the polymer is a shear thinning fluid or a shear thickening fluid. How would you expect $P_1(t)$ for these types of material to compare to that for a Newtonian fluid?

The geometry of the problem is illustrated in Figure 2.14. A cylindrical

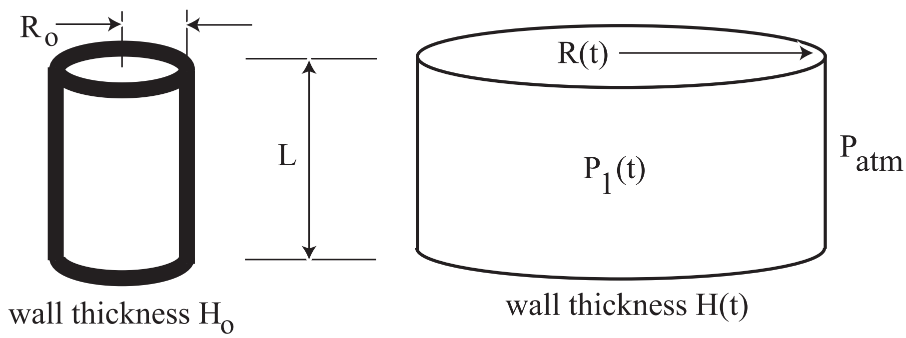

Figure 2.13: Idealization of the blow molding process as the inflation of a cylindrical parison of polymer melt.

coordinate system is used in order to take advantage of the symmetry of the problem. On the basis of the assumptions and symmetry we may immediately conclude: $v_z = 0$ and $v_\theta = 0$. There is no θ or z dependence of any variables. We are thus interested in determining within the polymer melt the variables: $v_r = v_r(r,t)$ and $P = P(r,t)$.

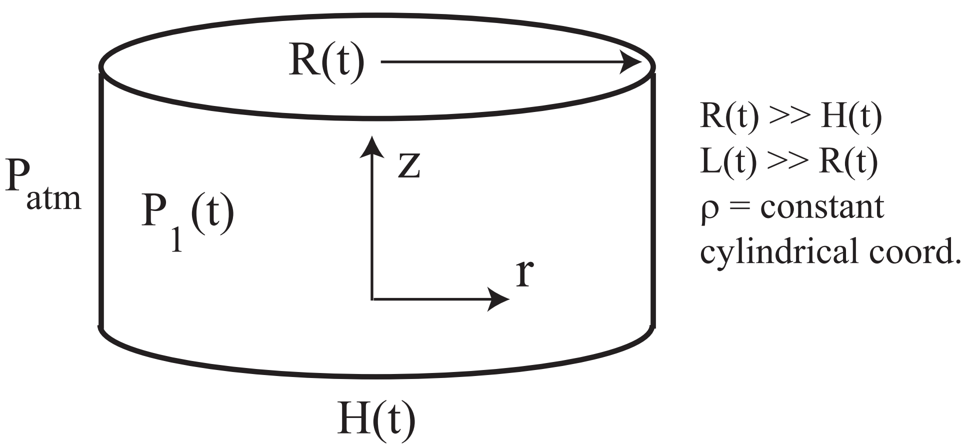

Figure 2.14: Sketch of problem geometry.

A constant inflation velocity means that: $\frac{dR(t)}{dt} = V_0 = constant$, which can be integrated to obtain: $R = R_0 + V_0 t$. Since the polymer is incompressible, the total volume of the cylindrical shell must remain constant. Using $H(t) \ll R(t)$ we may estimate: $Volume \cong 2\pi R_0 H_0 L = 2\pi R(t) H(t) L$, yielding

$$H(t) = H_0 \frac{R_0}{R_0 + V_0 t} \ . \tag{2.66}$$

The equation of continuity in cylindrical coordinates for an incompressible fluid with zero θ and z velocities reduces to:

$$\frac{\partial}{\partial r}(r v_r) + \frac{1}{r} = 0 \ . \tag{2.67}$$

This can be immediately integrated to obtain: $v_r = \frac{C(t)}{r}$ for some function $C(t)$. (Why is the "constant" of integration potentially a function of time?) The boundary condition at the inner radius is:

$$R(t) v_r|_{R(t)} = R(t)\frac{dR(t)}{dt} = R(t) V_0 \ . \tag{2.68}$$

Application of this boundary condition yields:

$$v_r = V_0 \frac{R_0 + V_0 t}{r} \ . \tag{2.69}$$

The velocity gradient tensor for the current flow is:

$$\underline{\nabla}\,\underline{\mathbf{v}} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & 0 & 0 \\ 0 & \frac{v_r}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Substitution of the velocity field obtained above yields the rate of deformation tensor:

$$\underline{\underline{\dot{\gamma}}} = \begin{bmatrix} -2V_0 \frac{R_0 + V_0 t}{r^2} & 0 & 0 \\ 0 & 2V_0 \frac{R_0 + V_0 t}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

This is an extensional flow since the only nonzero components of the rate of deformation tensor are along the diagonal. The scalar shear rate is:

$$\dot{\gamma} = 2V_0 \frac{R_0 + V_0 t}{r^2} \ . \tag{2.70}$$

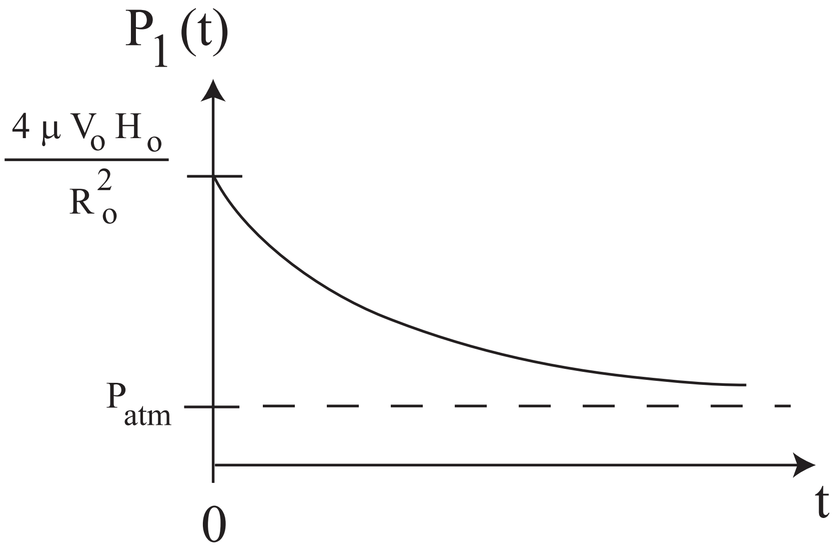

Figure 2.15: Blow pressure as a function of time.

Assuming that the fluid is Newtonian with a constant viscosity μ and that the inertial terms are negligible, the equation of motion in cylindrical coordinates for a fluid of constant density reduces in this case to:

$$0 = -\frac{\partial P}{\partial r} + \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) .$$

However, from Equation 2.69 we know that $\frac{\partial}{\partial r}(r v_r) = 0$ and thus $\frac{\partial P}{\partial r} = 0$ and the pressure within the polymer is a function of t only, $P = P(t)$.

The constitutive equation for a Newtonian fluid has been given previously and based on the results above, we can immediately write the total stress tensor:

$$\underline{\underline{\pi}} = \begin{bmatrix} P(t) + 2\mu V_0 \frac{R_0+V_0 t}{r^2} & 0 & 0 \\ 0 & P(t) - 2\mu V_0 \frac{R_0+V_0 t}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Since interfacial tension and inertial effects are negligible the air pressures inside and outside the shell are balanced by the stresses within the polymer melt. We are thus interested in the stresses on the inside and outside surfaces of the expanding shell, which can be obtained by selecting the appropriate components of the total stress tensor. At the inner surface we obtain:

$$P_1(t) = P(t) + 2\mu V_0 \frac{R_0 + V_0 t}{R^2(t)}$$

while at the outer surface

$$P_{atm}(t) = P(t) + 2\mu V_0 \frac{R_0 + V_0 t}{[R(t) + H(t)]^2} \, .$$

Subtracting these and recognizing that $H(t) \ll R(t)$ leads to:

$$P_1(t) - P_{atm} = \frac{4\mu V_0 H(t)}{R^2(t)} = \frac{4\mu V_0 H_0 R_0}{[R_0 + V_0 t]^3} \, . \tag{2.71}$$

The required blowing pressure as a function of time is plotted in Figure 2.15.

For non-Newtonian fluids the shear rate still changes according to Equation 2.70 (Why?). Thus, the deformation rate decreases approximately as $1/R$. As a function of shear rate, three fluids of the same zero shear viscosity will qualitatively behave as illustrated in Figure 2.16. The resulting pressure effects are shown in Figure 2.17.

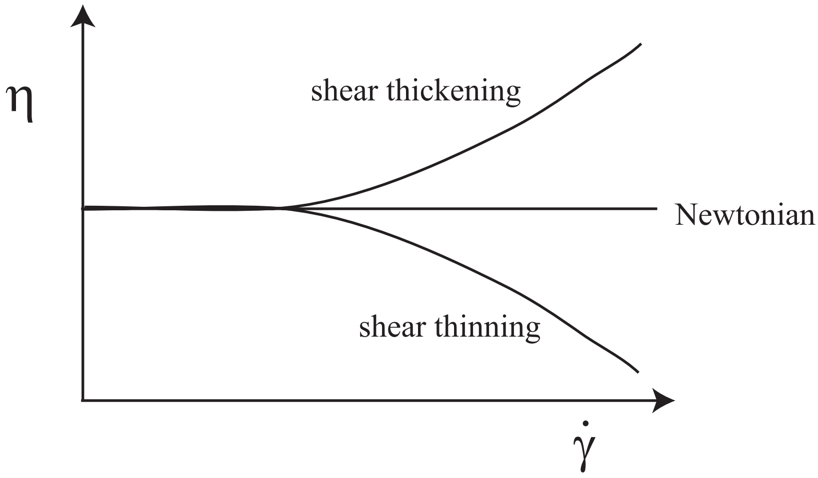

Figure 2.16: Qualitative behavior of three fluids with the same zero shear viscosity.

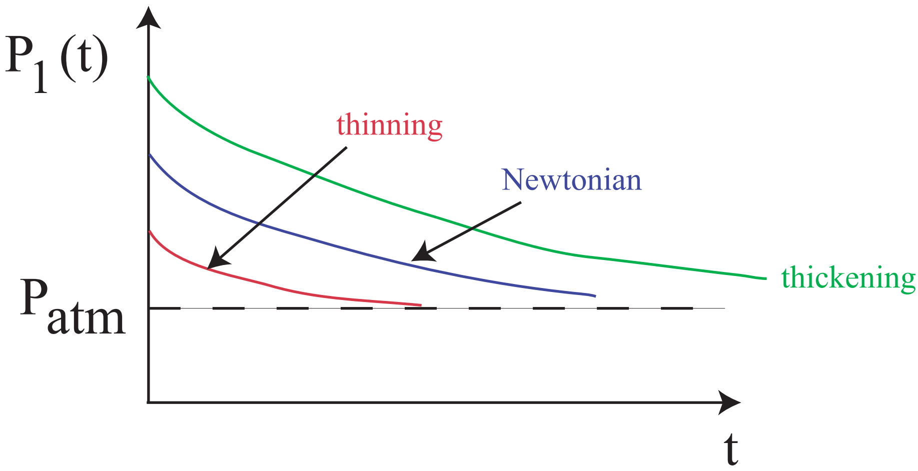

Figure 2.17: Qualitative differences in the required blowing pressures anticipated for different types of fluids.

2.3.6 Problems

1. **The Rate of Deformation Tensor.** For *any* arbitrary deformation of an incompressible fluid, what is the sum $\dot{\gamma}_{11} + \dot{\gamma}_{22} + \dot{\gamma}_{33}$?

2. **Wire Coating: Axial Drag Flow between Concentric Cylinders.** The manufacture of coated wire involves drawing of the wire through a cylindrical die, Figure 2.18. Polymer melt from a reservoir coats the wire as it is drawn out of the die. Consider the flow created in the space formed by two concentric cylinders of radii R_o and R_i, by the inner cylinder moving with an axial velocity V. The length $L \gg R$. The system is open to the atmosphere at both ends, $P_1 = P_2 = P_{atm}$. Assume well-developed isothermal flow of an incompressible Newtonian fluid. Neglect the effect of gravity.

   (a) Draw the flow channel, select an appropriate coordinate system, visualize the flow on physical grounds, and draw conclusions concerning the velocity components on the basis of symmetry and these physical grounds.

   (b) Reduce the continuity equation to the form appropriate for the flow field.

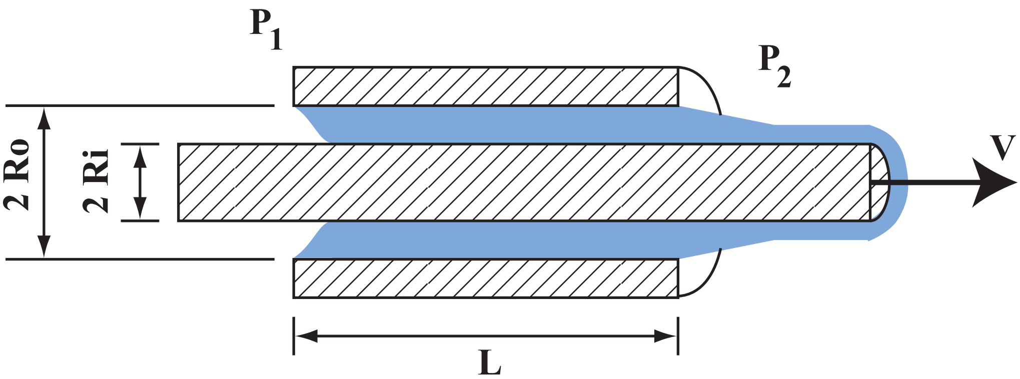

Figure 2.18: Cross-section of the wire coating process.

(c) Reduce the equation of motion to the form appropriate for the flow field.

(d) State the boundary conditions in physical and mathematical terms.

(e) Solve for the velocity profile within the flow channel. Sketch the velocity distribution.

(f) What is the thickness of the coating produced on the central cylinder (the wire)?

(g) What is the force required to pull the central cylinder at velocity V? (The effects of interfacial tension are negligible.)

(h) Rework the problem for the case of $P_1 \neq P_2$.

(i) Rework the problem for the case where the inner cylinder is rotated at a constant angular velocity Ω_0.

(j) Rework the problem for the case where the fluid is a power law fluid.

3. **Uniaxial Stretching of a Cylinder.** Consider the uniaxial stretching of a cylinder of polymer melt, as illustrated in Figure 2.19. Assume that the material maintains its cylindrical shape as it is stretched at velocity U and that the radius $R(t)$ is independent of z. Also assume that the fluid is incompressible and that the influence of gravity is negligible.

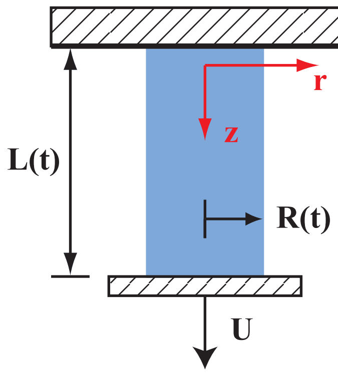

Figure 2.19: Stretching of a cylinder of fluid.

(a) Using the continuity equation, show that the velocity field within the stretching cylinder is:

$$v_r = -\frac{Ur}{2L(t)} \quad ; \quad v_z = \frac{Uz}{L(t)}$$

What is R(t)?

(b) Give the components of $\underline{\underline{\dot{\gamma}}}$ for this flow. What kind of flow field is this?

(c) Neglecting surface tension and inertial effects, calculate the force F required to pull the cylinder for a Newtonian fluid of viscosity μ.

(d) If instead, F is a constant (and thus U is a function of time), how does L change with time?

(e) How must L be programmed, i.e. give $L(t)$, such that the components of $\underline{\underline{\dot{\gamma}}}$ are independent of time. Is this result true only for Newtonian fluids? Why or why not?

(f) Neglecting surface tension and inertial effects, calculate the force $F(t)$ required to pull the cylinder at constant velocity U for a power law fluid of power law exponent n and prefactor m. If instead F is constant, how does L change with time?

4. **Biaxial Stretching of a Sheet.** Biaxial orientation of polymer sheet or film is commonly used to improve mechanical properties. Consider

the biaxial stretching of a sheet of polymer melt, as shown in Figure 2.20. The sheet is initially square, with the same width and length, L_0, and a thickness of H_0. Starting at time zero, a series of travelling clamps on all four edges of the sheet simultaneously stretch the sheet at a constant velocity V in the x and y directions. The sheet is free to deform in the thickness direction, which is open to the atmosphere. The sheet maintains its square shape as it is deformed, with width and length $L(t)$ and thickness $H(t)$. Assume that the material is incompressible and that the influence of gravity is negligible.

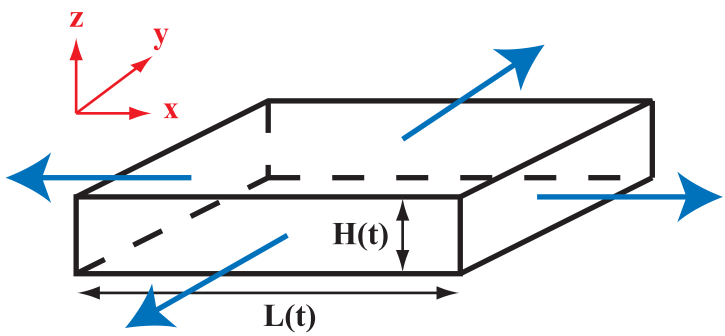

Figure 2.20: Simultaneous stretching of a square sheet in two directions.

(a) Using the equation of continuity, show that the velocity field within the stretching sheet is:

$$v_x = 2V\frac{x}{L(t)} \; ; \; v_y = 2V\frac{y}{L(t)} \; ; \; v_z = -4V\frac{z}{L(t)}$$

What is H(t)?

(b) Give the components of $\dot{\underline{\underline{\gamma}}}$ for this flow. What kind of flow field is this?

(c) Neglecting surface tension and inertial effects, calculate the force $F(t)$ that the clamps must exert on each edge of the sheet in order to impose the specified deformation if the polymer melt is a Newtonian fluid of constant viscosity μ.

(d) If instead F is held constant at F_0 (and thus V is a function of time), how does H change with time?

(e) How must V be programmed, i.e. give $V(t)$ such that the components of $\underline{\underline{\dot{\gamma}}}$ are independent of time? Is this result true only for Newtonian fluids? Why or why not?

(f) Neglecting surface tension and inertial effects, calculate the force $F(t)$ required to stretch the sheet at constant velocity V for a power law fluid of power law exponent n and prefactor m. If instead F is a constant, how does H change with time?

2.3.7 Further Reading

Excellent derivations and explanations of the equations of continuity and motion may be found in:

- R.B. Bird, W.E. Stewart, E.N. Lightfoot **Transport Phenomena**, John Wiley and Sons, New York (1960).
- R.B. Bird, R.C. Armstrong, O. Hassager, **Dynamics of Polymeric Liquids, Vol. 1**, John Wiley and Sons, New York (1987).

Numerous illustrative problems, some with solutions may be found in:

- W.F. Hughes, J.A. Brighton, **Fluid Dynamics**, Schaum's Outline Series, McGraw-Hill Book Company (1967).
- G.E. Mase, **Continuum Mechanics**, Schaum's Outline Series, McGraw-Hill Book Company (1970).

2.3.8 Expanded forms of the Equation of Motion

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial P}{\partial x}-\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+\rho g_x \tag{2.72}$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial P}{\partial y}-\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)+\rho g_y \tag{2.73}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}-\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z \tag{2.74}$$

Table 2.3: The equation of motion in rectangular coordinates.

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)= \\ -\frac{\partial P}{\partial r}-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}-\frac{\tau_{\theta\theta}}{r}+\frac{\partial \tau_{rz}}{\partial z}\right)+\rho g_r \qquad (2.75)$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)= \\ -\frac{1}{r}\frac{\partial P}{\partial \theta}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}\right)+\rho g_\theta \qquad (2.76)$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)= \\ -\frac{\partial P}{\partial z}-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z \qquad (2.77)$$

Table 2.4: The equation of motion in cylindrical coordinates.

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)=$$
$$-\frac{\partial P}{\partial r}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{r\theta}\sin\theta)\right.$$
$$\left.+\frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial\phi}-\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right)+\rho g_r \tag{2.78}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)=$$
$$-\frac{1}{r}\frac{\partial P}{\partial \theta}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta\theta}\sin\theta)\right.$$
$$\left.+\frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial\phi}+\frac{\tau_{r\theta}}{r}-\frac{\cot\theta}{r}\tau_{\phi\phi}\right)+\rho g_\theta \tag{2.79}$$

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta\right)=$$
$$-\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi})+\frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial\phi}\right.$$
$$\left.+\frac{\tau_{r\phi}}{r}+\frac{2\cot\theta}{r}\tau_{\theta\phi}\right)+\rho g_\phi \tag{2.80}$$

Table 2.5: The equation of motion in spherical coordinates.

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x \tag{2.81}$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y \tag{2.82}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z \tag{2.83}$$

Table 2.6: The Navier-Stokes equation in rectangular coordinates, for fluids of *constant density* ρ and *constant viscosity* μ.

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=$$
$$-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]+\rho g_r \quad (2.84)$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=$$
$$-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}\right.$$
$$\left.+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]+\rho g_\theta \quad (2.85)$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=$$
$$-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho g_z \quad (2.86)$$

Table 2.7: The Navier-Stokes equation in cylindrical coordinates, for fluids of *constant density* ρ and *constant viscosity* μ.

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)=$$
$$-\frac{\partial P}{\partial r}+\mu\left(\underline{\nabla}^2 v_r-\frac{2}{r^2}v_r-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2}{r^2}v_\theta\cot\theta\right.$$
$$\left.-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_r \tag{2.87}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)=$$
$$-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\underline{\nabla}^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}\right.$$
$$\left.-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_\theta \tag{2.88}$$

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta\right)=$$
$$-\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}+\mu\left(\underline{\nabla}^2 v_\phi-\frac{v_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}\right.$$
$$\left.+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right)+\rho g_\phi \tag{2.89}$$

Table 2.8: The Navier-Stokes equation in spherical coordinates, for fluids of *constant density* ρ and *constant viscosity* μ.